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ADDENDUM

Transformation formulae of the solutions of the two-dimensional Toda lattice

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Abstract. Two transformation formulae of the solutions of the two-dimensional Toda lattice are presented. The first formula, which can be obtained by the Zakharov-Shabat dressing method, is proved by applying the Hirota bilinear transformation method and gives the basis for the second new formula.

In the development of soliton theory, various exact methods have been found for solving non-linear evolution equations. Among them, the bilinear transformation method [1], initiated by Hirota, is a powerful tool. In the bilinear formalism, a given non-linear equation is first transformed into a bilinear form through a dependent variable transformation. Then the multi-soliton solutions, the Bäcklund transformations and an infinite number of conservation laws of this bilinear equation can be derived in a systematic way.

Recently Freeman and Nimmo [2, 3], Hirota [4, 5], Matsuno [6, 7], Nakamura [8, 9], Sachs [10], Yuasa [11] and the author [12, 13] have found that in the bilinear formalism the multi-soliton solutions, the rational solutions and the non-linear superposition formulae can be proved by purely algebraic procedures and the knowledge of the inverse scattering transform method is not required.

The main purpose of this addendum is to present the direct proof of a transformation formula which can be obtained by the Zakharov-Shabat dressing method [14]. Then a new generalisation is derived by making use of this result.

We have the two-dimensional Toda lattice in bilinear form:

$$f_n(\partial_x\partial_s f_n)-(\partial_x f_n)(\partial_s f_n)-f_{n+1}f_{n-1}+f_n^2=0 \tag{1.1}$$

and its Bäcklund transformation:

$$\begin{aligned}\partial_x\tilde{f}_n&=\frac{g_n}{g_{n+1}}\tilde{f}_{n+1}+\frac{\partial_x g_{n+1}}{g_{n+1}}\tilde{f}_n\\ \partial_s\tilde{f}_n&=-\frac{g_{n+1}}{g_n}\tilde{f}_{n-1}+\frac{\partial_s g_n}{g_n}\tilde{f}_n.\end{aligned} \tag{1.2}$$

Since (1.1) is invariant under the variable transformations $x\to -x$, $s\to -s$, $m\to 2n-m$, we can get another Bäcklund transformation of (1.1)

$$\begin{aligned}\partial_x\bar{f}_n&=-\frac{g_n}{g_{n-1}}\bar{f}_{n-1}+\frac{\partial_x g_{n-1}}{g_{n-1}}\bar{f}_n\\ \partial_s\bar{f}_n&=\frac{g_{n-1}}{g_n}\bar{f}_{n+1}+\frac{\partial_s g_n}{g_n}\bar{f}_n.\end{aligned} \tag{1.3}$$

Theorem 1. If g_n satisfies the two-dimensional Toda lattice (1.1), $\tilde{f}^i_n$ $(1 \leq i \leq N)$ satisfy (1.2) and $\bar{f}^i_n$ $(1 \leq i \leq N)$ satisfy (1.3), then $f_n = g_{n+1} \det H$ also satisfies (1.1), where

$$(H)_{ij} = h_{ij}(n) = c_{ij} + \sum_{m=-\infty}^{n} \frac{\tilde{f}^j_m \bar{f}^i_{m+1}}{g_m g_{m+1}} \qquad 1 \leq i, j \leq N.$$

Proof. For simplicity, we introduce new dependent variables φ^i_n, $\bar{\varphi}^i_n$ $(1 \leq i \leq N)$ by $\varphi^i_n = \tilde{f}^i_n / g_{n+1}$ and $\bar{\varphi}^i_n = \bar{f}^i_{n+1}/g_n$, then we have

$$\partial_x \varphi^i_n = u_n \varphi^i_{n+1} \qquad \partial_s \varphi^i_n = -\varphi^i_{n-1} + v_n \varphi^i_n \tag{2}$$

$$\partial_x \bar{\varphi}^i_n = -u_{n-1} \bar{\varphi}^i_{n-1} \qquad \partial_s \bar{\varphi}^i_n = \bar{\varphi}^i_{n+1} - v_n \bar{\varphi}^i_n \tag{3}$$

where

$$u_n = \frac{g_n g_{n+2}}{g^2_{n+1}} \qquad v_n = \frac{\partial_s g_n}{g_n} - \frac{\partial_s g_{n+1}}{g_{n+1}}$$

and

$$(H)_{ij} = h_{ij}(n) = c_{ij} + \sum_{m=-\infty}^{n} \bar{\varphi}^i_m \varphi^j_m \qquad 1 \leq i, j \leq N.$$

Using the ordinary matrix properties, we have

$$\begin{aligned} f_{n+1} &= g_{n+2} \det(h_{ij}(n+1)) \\ &= g_{n+2} \det(h_{ij}(n) + \bar{\varphi}^i_{n+1} \varphi^j_{n+1}) \\ &= g_{n+2}\left(\det H + \sum_{k=1}^{N} \det A^1_k\right) \end{aligned} \tag{4}$$

where

$$(A^1_k)_{ik} = \begin{cases} h_{ij}(n) & i \neq k \qquad (5a) \\ \bar{\varphi}^i_{n+1} \varphi^j_{n+1} & i = k. \qquad (5b) \end{cases}$$

At the first stage of the above expression, instead of (5*b*) we have a slightly different form $(A^1_k)_{ij} = h_{ij}(n+1)$ $(k < i \leq N)$. Then in the det A^1_k we have added the kth row multiplied by the factor $-\bar{\varphi}^i_{n+1}/\bar{\varphi}^k_{n+1}$ to the ith row for $k < i \leq N$. This gives the expression of (5*b*).

Furthermore, if we use the matrix identity

$$\sum_{k=1}^{N} \det A_k = - \begin{vmatrix} a_{11} & a_{12} & \cdots & x_{1N} & x_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} & x_N \\ y_1 & y_2 & \cdots & y_N & 0 \end{vmatrix}$$

where

$$(A_k)_{ij} = \begin{cases} a_{ij} & i \neq k \\ x_i y_j & i = k \end{cases}$$

then (4) becomes

$$f_{n+1} = g_{n+2}\left(\det H - \begin{vmatrix} & H & & \bar{\varphi}^N_{n+1} \\ & & & \vdots \\ & & & \bar{\varphi}^N_{n+1} \\ \varphi^1_{n+1} & \cdots & \varphi^N_{n+1} & 0 \end{vmatrix}\right) \tag{6}$$

similarly, we have

$$f_{n-1} = g_n\left(\det H + \sum_{k=1}^{N} \det A^2_k\right)$$

where

$$(A^2_k)_{ij} = \begin{cases} h_{ij}(n) & i \neq k \\ -\bar{\varphi}^i_n \varphi^j_n & i = k \end{cases}$$

and

$$f_{n-1} = g_n\left(\det H + \begin{vmatrix} & H & & \bar{\varphi}^1_n \\ & & & \vdots \\ & & & \bar{\varphi}^N_n \\ \varphi^1_n & \cdots & \varphi^N_n & 0 \end{vmatrix}\right). \tag{7}$$

In order to calculate the quantities including differential operators, we see the operations of ∂_x, ∂_s upon the matrix element $h_{ij}(n)$. We have

$$\begin{aligned} \partial_x h_{ij}(n) &= \sum_{m=-\infty}^{n} [(\partial_x \bar{\varphi}^i_m)\varphi^j_m + \bar{\varphi}^i_m(\partial_x \varphi^j_m)] \\ &= \sum_{m=-\infty}^{n} (-u_{m-1}\bar{\varphi}^i_{m-1}\varphi^j_m + u_m \bar{\varphi}^i_m \varphi^j_{m+1}) \\ &= u_n \bar{\varphi}^i_n \varphi^j_{n+1} \\ \partial_s h_{ij}(n) &= \sum_{m=-\infty}^{n} [(\partial_s \bar{\varphi}^i_m)\varphi^j_m + \bar{\varphi}^i_m(\partial_s \varphi^j_m)] \\ &= \sum_{m=-\infty}^{n} [(\bar{\varphi}^i_{m+1} - v_m \bar{\varphi}^i_m)\varphi^j_m - \bar{\varphi}^i_m(\varphi^j_{m-1} - v_m \varphi^j_m)] \\ &= \bar{\varphi}^i_{n+1}\varphi^j_n \\ \partial_x \partial_s h_{ij}(n) &= (\partial_x \bar{\varphi}^i_{n+1})\varphi^j_n + \bar{\varphi}^i_{n+1}(\partial_x \varphi^j_n) \\ &= u_n(-\bar{\varphi}^i_n \varphi^j_n + \bar{\varphi}^i_{n+1}\varphi^j_{n+1}). \end{aligned}$$

Now we calculate the quantities $\partial_x \det H$, $\partial_s \det H$ and $\partial_x \partial_s \det H$:

$$\partial_x \det H = \sum_{k=1}^{N} \det A^3_k$$

where

$$(A^3_k)_{ij} = \begin{cases} h_{ij}(n) & i \neq k \\ \partial_x h_{ij}(n) = u_n \bar{\varphi}^i_n \varphi^j_{n+1} & i = k. \end{cases}$$

Thus we have

$$\partial_x \det H = -u_n \begin{vmatrix} & & \bar\varphi_n^1 \\ & H & \vdots \\ & & \bar\varphi_n^N \\ \varphi_{n+1}^1 & \cdots \ \varphi_{n+1}^N & 0 \end{vmatrix} \tag{8}$$

$$\partial_s \det H = \sum_{k=1}^{N} \det A_k^4$$

where

$$(A_k^4)_{ij} = \begin{cases} h_{ij}(n) & i \neq k \\ \partial_s h_{ij}(n) = \bar\varphi_{n+1}^i \varphi_n^j & i = k. \end{cases}$$

Thus we have

$$\partial_s \det H = -\begin{vmatrix} & & \bar\varphi_{n+1}^1 \\ & H & \vdots \\ & & \bar\varphi_{n+1}^N \\ \varphi_n^1 & \cdots \ \varphi_n^M & 0 \end{vmatrix} \tag{9}$$

$$\partial_x \partial_s \det H = \sum_{k=1}^{N} \det A_k^5 + \sum_{\substack{k,k'=1 \\ k\neq k}}^{N} \det A_{kk'} \tag{10}$$

where

$$(A_k^5)_{ij} = \begin{cases} h_{ij}(n) & i \neq k \\ \partial_x \partial_s h_{ij}(n) = u_n(-\bar\varphi_n^i \varphi_n^j + \bar\varphi_{n+1}^i \varphi_{n+1}^j) & i = k \end{cases}$$

$$(A_{kk'})_{ij} = \begin{cases} h_{ij}(n) & i \neq k, k' \\ \partial_x h_{ij}(n) = u_n \bar\varphi_n^i \varphi_{n+1}^j & i = k \\ \partial_s h_{ij}(n) = \bar\varphi_{n+1}^i \varphi_n^j & i = k'. \end{cases}$$

Thus we have

$$\sum_{k=1}^{N} \det A_k^5 = u_n \sum_{k=1}^{N} (\det A_k^1 + \det A_k^2)$$

$$= u_n \left(-\begin{vmatrix} & & \bar\varphi_{n+1}^1 \\ & H & \vdots \\ & & \bar\varphi_{n+1}^N \\ \varphi_{n+1}^1 & \cdots \ \varphi_{n+1}^N & 0 \end{vmatrix} + \begin{vmatrix} & & \bar\varphi_n^1 \\ & H & \vdots \\ & & \bar\varphi_n^N \\ \varphi_n^1 & \cdots \ \varphi_n^N & 0 \end{vmatrix} \right) \tag{11}$$

Applying the Laplace expansion theorem, we can obtain

$$\det A_{kk'} = \sum_{\substack{l,l'=1 \\ l<l'}}^{N} (-1)^{k+k'+l+l'} \operatorname{sgn}(k'-k) \begin{vmatrix} (A_{kk'})_{kl} & (A_{kk'})_{kl'} \\ (A_{kk'})_{k'l} & (A_{kk'})_{k'l'} \end{vmatrix} H\begin{pmatrix} kk' \\ l\,l' \end{pmatrix}$$

$$= u_n \sum_{\substack{l,l'=1 \\ l<l'}}^{N} (-1)^{k+k'+l+l'} \operatorname{sgn}(k'-k) \bar\varphi_n^k \bar\varphi_{n+1}^{k'} \begin{vmatrix} \varphi_{n+1}^1 & \varphi_{n+1}^{l'} \\ \varphi_n^1 & \varphi_n^{l'} \end{vmatrix} H\begin{pmatrix} k & k' \\ l & l' \end{pmatrix}$$

where $H\binom{k\ k'}{l\ l'}$ is the same as det H except the kth and k'th rows and the lth and l'th columns are deleted from it. Thus we have

$$\sum_{\substack{k,k'=1\\k\neq k'}}^{N} \det A_{kk'} = \sum_{\substack{k,k'=1\\k<k'}}^{N} (\det A_{kk'} + \det A_{k'k})$$

$$= -u_n \sum_{\substack{k,k'=1\\k<k'}}^{N} \sum_{\substack{l,l'=1\\l<l'}}^{N} (-1)^{k+k'+l+l'} \begin{vmatrix} \bar\varphi_n^k & \bar\varphi_{n+1}^k \\ \bar\varphi_n^{k'} & \bar\varphi_{n+1}^{k'} \end{vmatrix} \begin{vmatrix} \varphi_n^l & \varphi_n^{l'} \\ \varphi_{n+1}^l & \varphi_{n+1}^{l'} \end{vmatrix} H\begin{pmatrix} k & k' \\ l & l' \end{pmatrix}$$

$$= -u_n \begin{vmatrix} & & & \bar\varphi_n^1 & \bar\varphi_{n+1}^1 \\ & H & & \vdots & \vdots \\ & & & \bar\varphi_n^N & \bar\varphi_{n+1}^N \\ \varphi_n^1 & \cdots & \varphi_n^N & 0 & 0 \\ \varphi_{n+1}^1 & \cdots & \varphi_{n+1}^N & 0 & 0 \end{vmatrix}. \tag{12}$$

From (6)-(12), we have

$$\begin{aligned}
&f_n(\partial_x\partial_s f_n) - (\partial_x f_n)(\partial_s f_n) - f_{n+1}f_{n-1} + f_n^2 \\
&\quad = g_{n+1}(\det H)[(\partial_x\partial_s g_{n+1})(\det H) + (\partial_x g_{n+1})(\partial_s \det H) + (\partial_s g_{n+1})(\partial_x \det H) \\
&\qquad + g_{n+1}(\partial_x\partial_s \det H)] - [(\partial_x g_{n+1})(\det H) + g_{n+1}(\partial_x \det H)] \\
&\qquad \times [(\partial_s g_{n+1})(\det H) + g_{n+1}(\partial_s \det H)] - f_{n+1}f_{n-1} + g_{n+1}^2(\det H)^2 \\
&\quad = [g_{n+1}(\partial_x\partial_s g_{n+1}) - (\partial_x g_{n+1})(\partial_s g_{n+1}) - g_{n+2}g_n + g_{n+1}^2](\det H)^2 \\
&\qquad + g_{n+2}g_n \left(\det H \begin{vmatrix} & & & \bar\varphi_n^1 & \bar\varphi_{n+1}^1 \\ & H & & \vdots & \vdots \\ & & & \bar\varphi_n^N & \bar\varphi_{n+1}^N \\ \varphi_n^1 & \cdots & \varphi_n^N & 0 & 0 \\ \varphi_{n+1}^1 & \cdots & \varphi_{n+1}^N & 0 & 0 \end{vmatrix} \right. \\
&\qquad - \begin{vmatrix} & H & & \bar\varphi_n^1 \\ & & & \vdots \\ & & & \bar\varphi_n^N \\ \varphi_{n+1}^1 & \cdots & \varphi_{n+1}^N & 0 \end{vmatrix} \begin{vmatrix} & H & & \bar\varphi_{n+1}^1 \\ & & & \vdots \\ & & & \bar\varphi_{n+1}^N \\ \varphi_n^1 & \cdots & \varphi_n^N & 0 \end{vmatrix} \\
&\qquad \left. + \begin{vmatrix} & H & & \bar\varphi_{n+1}^1 \\ & & & \vdots \\ & & & \bar\varphi_{n+1}^N \\ \varphi_{n+1}^1 & \cdots & \varphi_{n+1}^N & 0 \end{vmatrix} \begin{vmatrix} & H & & \bar\varphi_n^1 \\ & & & \vdots \\ & & & \bar\varphi_n^N \\ \varphi_n^1 & \cdots & \varphi_n^N & 0 \end{vmatrix} \right) = 0.
\end{aligned} \tag{13}$$

Equation (13) holds due to Jacobi's formula. Therefore we have proved that f_n actually satisfies the bilinear equation (1.1).

If we take $g_n = 1$, then we get the cylindrical N-soluton solutions of (1.1) [14]

$$f_n = \left(c_{ij} + \sum_{m=-\infty}^{n} \bar\varphi_m^i \varphi_m^j \right) \qquad 1 \leq i, j \leq N$$

where the quantities φ_n^i, $\bar\varphi_n^i$ $(1 \leq i, j \leq N)$ satisfy

$$\partial_x \varphi_n^i = \varphi_{n+1}^i \qquad \partial_s \varphi_{n-1}^i = -\varphi_{n-1}^i$$

$$\partial_x \bar\varphi_n^i = -\bar\varphi_{n-1}^i \qquad \partial_s \bar\varphi_n^i = \bar\varphi_{n+1}^i.$$

We can generalise theorem 1 for $N = \infty$.

Theorem 2.

$$f_n = g_{n+1}\left(1 + \sum_{m=1}^{\infty} \varepsilon^m \sum_{\substack{i_1, i_2, \ldots, i_m = 1 \\ i_1 < i_2 < \ldots < i_m}}^{\infty} \bar{g}_{i_1 i_2 \ldots i_m}\right)$$

satisfies (1.1), where

$$\bar{g}_{i_1 i_2 \ldots i_m} = \begin{vmatrix} h^0_{i_1 i_1}(n) & \cdots & h^0_{i_1 i_m}(n) \\ \vdots & \ddots & \vdots \\ h^0_{i_m i_1}(n) & \cdots & h^0_{i_m i_m}(n) \end{vmatrix}$$

and

$$h^0_{ij}(n) = c_i \delta_{ij} + \sum_{m=-\infty}^{n} \frac{\tilde{f}^j_m \bar{f}^i_{m+1}}{g_m g_{m+1}}.$$

This result can be proved by Nakamura's method [8].

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